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Continuum description of rarefied gas dynamics. I. Derivation from kinetic theory

Xinzhong Chen
Department of Astronomy, Columbia University, New York, New York 10027

Hongling Rao
Microeletronics Sciences Laboratories, Columbia University, New York, New York 10027

Edward A. Spiegel
Department of Astronomy, Columbia University, New York, New York 10027

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We describe an asymptotic procedure for deriving continuum equations from the kinetic theory of a simple gas. As in the works of Hilbert, of Chapman, and of Enskog, we expand in the mean flight time of the constituent particles of the gas, but we do not adopt the Chapman-Enskog device of simplifying the formulas at each order by using results from previous orders. In this way, we are able to derive a new set of fluid dynamical equations from kinetic theory, as we illustrate here for the relaxation model for monatomic gases. We obtain a stress tensor that contains a dynamical pressure term (or bulk viscosity) that is process dependent and our heat current depends on the gradients of both temperature and density. On account of these features, the equations apply to a greater range of Knudsen number (the ratio of mean free path to macroscopic scale) than do the Navier-Stokes equations, as we see in the accompanying paper. In the limit of vanishing Knudsen number, our equations reduce to the usual Navier-Stokes equations with no bulk viscosity.

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I. INTRODUCTION

The derivation of fluid equations from kinetic theory is often effected by an expansion of the solution of the kinetic equation in the Knudsen number, the ratio of the mean free path to the characteristic macroscopic scale. The method generally in use was developed by Chapman and Enskog [1] whose aim in part was to remedy deficiencies in the earlier work of Hilbert [2,3]. Chapman and Enskog derived the Euler equations at zeroth order and the Navier-Stokes (NS) equations at first order, starting from the Boltzmann equation of kinetic theory. But the Navier-Stokes (NS) equations do not adequately describe the dynamics of fluids when the Knudsen number is not very small. The inadequacies of the NS equations in dealing with the problems of rarefied media are well documented [4] and we shall cite empirical evidence for this when we compare our theory with experiment in Paper II [28] of this series. Problems arise, for example, in the study of shock waves since the thickness of a shock wave is generally of the order of the mean free path of the particles that make up the medium through which the shock propagates. However, the difficulty has little if anything to do with the nonlinearity of the shock waves, as we may see from the similar failure of the NS equations to predict accurately the propagation of linear sound waves when their periods are comparable to the mean flight times of the constituent particles (paper II).

To improve matters, one may try solving the kinetic equations directly [4], but this leads to a problem in higher dimensions than is encountered in solving the continuum fluid equations. Alternatively, one may seek to derive from the kinetic theory a set of fluid equations with a greater domain of validity than the NS equations. This was the aim of attempts to go to higher order in the development of the kinetic equation in Knudsen number. Higher order approximations, such as the Burnett or super-Burnett developments [5,6] [and that of Woods [7]] have hardly improved matters and the resulting equations do not work well when the Knudsen number is not infinitesimal [3]. Moreover, the continuum equations found in these higher approximations are very complicated so that they do not seem to repay the effort involved in their use.

Some hope for improvement of the situation was raised when Grad [8] introduced his moment method, whose leading order results are the Navier-Stokes equations. However, the further development of Grad's method [9,10] does not produce very rapid convergence nor does it easily give very accurate solutions of the problems mentioned here (as we shall see in Paper II). Levermore [11] has proposed an alternate moment method that avoids the failures of causality that sometimes arise with the Grad's method. The newer method has produced interesting results on shock structure [12] but as yet has not given results with heat flux.

Other approaches that are being explored are the resummation of the terms of the Chapman-Enskog expansion [13–15] and the flux-limited diffusion theory [16]; both of these approaches have proved useful in radiative transfer theory (the former in Refs. [17,18] and the latter in Ref. [19]). A promising and interesting approach is by way of thermodynamic (or phenomenological) models [9,10] that give good agreement with experiment [9]. The relativistic extensions of these latter procedures [20] produce hyperbolic systems and do not violate causality.

Here we propose a modification of the asymptotic procedure of Chapman and Enskog. We begin, as they do, with an expansion in mean free paths of the constituent particles, as did Hilbert. However, we do not simplify the results at a given order by introducing expressions (mainly for time derivatives of fluid quantities) from previous orders as did Chapman and Enskog. As we shall see, this seemingly small difference in the methods makes for large differences in the results when they are applied to cases with finite mean free paths. As mentioned in Ref. [21] and more fully documented in Paper II, even our first-order asymptotics provides acceptable accuracy when comparison is made with experimental results.

In this paper, we derive continuum equations from the relaxation model of kinetic theory [22–24]. In the relaxation model, the approach toward equilibrium takes place in a relaxation time τ, which is determined by the mean flight time of the constituent particles of the medium. So the analog of Hilbert's approach is here an expansion in relaxation time. In our version of this development, we shall not recycle the lower orders through the current order as is done in Chapman-Enskog theory. The fluid equations following from our procedure, generalize the Navier-Stokes equations, which we shall recover from ours by a simple development for the stress tensor and the heat flux in terms of τ, effectively the Knudsen number. In that sense, a byproduct of our development is a relatively simple derivation of the Navier-Stokes equations, one that avoids some of the complications of the Chapman-Enskog method.

II. SOME KINETIC THEORY

Consider a gas made up of identical particles of mass m obeying classical Hamiltonian dynamics. The phase space of a single particle is six dimensional and its coordinates are spatial position $\mathbf{x}$, and velocity $\mathbf{v}$. The expected number of particles in a phase volume $d\mathbf{x}d\mathbf{v}$ is $f(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v}$, where f is the density in phase space (or one-particle distribution function). For a Hamiltonian system the phase flow is incompressible in the sense that its six-dimensional velocity is solenoidal, and the relaxation model of kinetic theory [22,25,24] that we adopt here is

$$\mathcal{D}f=\frac{f_0-f}{\tau}, \tag{2.1}$$

with the following notation. The streaming operator is

$$\mathcal{D}=\partial_t+v^i\partial_{x^i}+a^i\partial_{v^i} \tag{2.2}$$

where $i=1,2,3$, repeated indices are summed, $\partial_{x^i}=\partial/\partial x^i$, $\partial_{v^i}=\partial/\partial v^i$, a^i represents the acceleration of a particle caused by an external force, and τ is the time scale on which the system relaxes to the equilibrium f_0.

While we prefer to think of Eq. (2.1) as a model, it may be also considered as an approximation to the Boltzmann equation where τ and f_0 are taken as approximations to functionals of f that appear in the Boltzmann collision operator. With either interpretation, the relaxation time τ is of the order of a collision time. Though τ might, in principle, depend on $\mathbf{v}$, we here assume that it depends on just the local values of the macroscopic fields. To give an expression for τ we then need to specify the macroscopic quantities.

The mass density of the fluid is defined as

$$\rho(\mathbf{x},t)=\int mf d\mathbf{v}, \tag{2.3}$$

where the integration is over all of velocity space. We may also introduce the particle number density $n=\rho/m$. A second important macroscopic quantity is the mean, or drift, velocity, defined as

$$\mathbf{u}(\mathbf{x},t)=\frac{1}{\rho}\int m\mathbf{v}f d\mathbf{v}. \tag{2.4}$$

We may then introduce the peculiar velocity $\mathbf{c}=\mathbf{v}-\mathbf{u}$ and use it to define the temperature as

$$T=\frac{1}{3R\rho}\int \mathbf{c}^2 f d\mathbf{v}, \tag{2.5}$$

where $R=k/m$ is the gas constant and k is the Boltzmann constant. We assume throughout this work that f goes rapidly to zero as $|\mathbf{v}|\to\infty$, so that f-weighted integrals over velocity space are finite and well defined.

The macroscopic quantities are all functions of $\mathbf{x}$ and t, and, when we speak of a temperature, we use the notion of local thermodynamic equilibrium, in which equilibrium formulas are used to describe nonequilibrium conditions locally in space and time. In the relaxation version of the kinetic equation, the interaction term on the right of Eq. (2.1), representing collisions amongst the particles, drives the system toward the equilibrium given by f_0, which we take to be the local Maxwell-Boltzmann distribution in the frame locally comoving with the matter, namely

$$f_0=n(2\pi RT)^{-3/2}\exp\left(-\frac{c^2}{2RT}\right). \tag{2.6}$$

This distribution depends on position and time only through its dependence on the local macroscopic quantities, n, $\mathbf{u}$, and T and it is, moreover, a local equilibrium solution of the Boltzmann equation.

To make the description work well, we try to arrange that the fields on which f_0 depends are most nearly those of the real flow. This ''osculating'' property of the assumed equilibrium may be imposed by matching conditions that are inherited from the theory of the Boltzmann equation. There the quantities that are conserved in two body collisions are the sums of the masses, the velocities, and the energies of the two colliding particles. We let ψ_α with $\alpha=0,1,\ldots,4$ be these summational invariants, that is $\psi_0=m$, $\psi_i=mv_i$, and $\psi_4=\frac{1}{2}m\mathbf{v}^2$ with $i=1,2,3$. In Boltzmann theory, this conservation property results in the orthogonality of the ψ_α with the collision term in the kinetic theory. This property is generally ascribed to kinetic models, including the relaxation model. Therefore when we multiply the right-hand side of equation by any of the ψ_α and integrate over velocity space, we get zero. This gives us the matching conditions

$$\int \psi_\alpha f_0 d\mathbf{v}=\int \psi_\alpha f d\mathbf{v}, \tag{2.7}$$

provided that we assume that τ does not depend on the velocity. This condition ensures that the true fluid fields are the ones appearing in Eq. (2.6).

Finally, we may specify that τ should be of the order of the mean flight time of the constituent particles, that is, the mean free path of particles divided by the mean speed. If the collision cross section of the particles is independent of velocity, then simple collision theory gives the result that $\tau \propto 1/\rho\sqrt{T}$, so that, generally speaking, τ is a function of the macroscopic variables.

III. EQUATIONS OF FLUID DYNAMICS

When we multiply Eq. (2.1) by $\tau\psi_\alpha$ and integrate over **v**, the right-hand side is zero because of the matching condition. If we add the assumption that τ does not depend on **v** we obtain these macroscopic equations:

$$\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0, \tag{3.1}$$

$$\rho(\partial_t\mathbf{u} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}) + \boldsymbol{\nabla}\cdot\mathsf{P} = \rho\mathbf{a}, \tag{3.2}$$

$$\tfrac{3}{2}\rho R(\partial_t T + \mathbf{u}\cdot\boldsymbol{\nabla}T) + \mathsf{P}{:}\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}\cdot\mathbf{Q} = 0, \tag{3.3}$$

where the colon stands for a double dot product and we have introduced the pressure tensor (or second moment)

$$\mathsf{P} = \int m\mathbf{c}\mathbf{c}f\,d\mathbf{v} \tag{3.4}$$

and the heat flux vector (a third moment) given by

$$\mathbf{Q} = \int \frac{1}{2}m\mathbf{c}^2\mathbf{c}f\,d\mathbf{v}. \tag{3.5}$$

In deriving these equations, we have performed integrations by parts under the integrations over velocity space. These give no boundary terms if, as we assume, f goes rapidly enough to zero as $|\mathbf{c}|$ gets large. [Strictly speaking, the P in Eq. (3.3) should be the transpose, but as the stress tensor is symmetric in this work, we need not indicate this.]

These macroscopic equations are a formal consequence of the kinetic equation (2.1). For them to be useful, we must supply adequate expressions for P and **Q**. The Navier-Stokes forms of these higher moments, which were first derived phenomenologically, were obtained for small mean free paths by the Chapman-Enskog method from the Boltzmann equation. The NS equations have also been extracted from Eq. (2.1) by this method [23]. Those developments follow on the work of Hilbert, who introduced a series expansion in mean free path for f into the Boltzmann equation [2,23,3]. In the case of the relaxation model, matters are simpler because it has an essentially linear form and this permits a clearer statement of the underlying approach to deriving the fluid equations. The procedure is to develop in τ, on the presumption that it is small compared to the macroscopic times that arise. Such an asymptotic development for the case of the relaxation model begins with an expansion of the form

$$f = \sum_{m=0}^{\infty} f_{(m)}\tau^m. \tag{3.6}$$

When we substitute Eq. (3.6) into Eq. (2.1) we obtain a series of equations for the $f_{(m)}$. The first of these is the anticipated condition $f_{(0)} = f_0$. The second approximation is simply

$$f_{(1)} = -\mathcal{D}f_0\,. \tag{3.7}$$

When we introduce expression (2.6) into this result we obtain the more explicit relation

$$f_{(1)} = -f_0\left[\mathcal{D}\ln\rho + \left(\frac{c^2}{2RT} - \frac{3}{2}\right)\mathcal{D}\ln T + \frac{1}{RT}\mathbf{c}\cdot\mathcal{D}\mathbf{c}\right]. \tag{3.8}$$

Matters are quite simple up to this point but when we go to higher orders, we encounter terms arising from the derivatives of τ. So we stop the expansion at this point since we already have enough in first order to obtain an interesting generalization of the equations of fluid mechanics.

To use the asymptotic results for f in Eqs. (3.4) and (3.5), we note that the higher moments can be written as

$$\mathsf{P} = \mathsf{P}_{(0)} + \tau\mathsf{P}_{(1)} + \cdots \tag{3.9}$$

and

$$\mathbf{Q} = \mathbf{Q}_{(0)} + \tau\,\mathbf{Q}_{(1)} + \cdots, \tag{3.10}$$

where

$$\mathsf{P}_{(n)} = \int m\mathbf{c}\mathbf{c}f_{(n)}d\mathbf{v} \quad \text{and} \quad \mathbf{Q}_{(n)} = \int \frac{1}{2}m\mathbf{c}^2\mathbf{c}f_{(n)}d\mathbf{v}. \tag{3.11}$$

Then we readily see that

$$\mathsf{P}_{(0)} = \int_0^\infty mc^4 f_0\,dc \int \mathbf{e}\,\mathbf{e}\,d\Omega, \tag{3.12}$$

where $c = |\mathbf{c}|$ and $\mathbf{e} = \mathbf{c}/c$.

With a little rearranging, the first of the two integrals in Eq. (3.12) becomes proportional to a gamma function. The second integral can be written as

$$\int \mathbf{e}\,\mathbf{e}\,d\Omega = \frac{4\pi}{3}\mathsf{I}, \tag{3.13}$$

where I is the unit tensor; its components are those of the Kronecker delta. Hence, we find that

$$\mathsf{P}_{(0)} = p\,\mathsf{I}, \tag{3.14}$$

where

$$p = R\rho T, \tag{3.15}$$

which we identify with the gas pressure. By contrast, the integral for the heat flux is odd in $\mathbf{c}$ and we find that $\mathbf{Q}_{(0)} = \mathbf{0}$. In short, the macroscopic equations at leading order are the Euler equations.

To evaluate $\mathsf{P}_{(1)}$, we note that

$$\mathcal{D} = \frac{D}{Dt} + \mathbf{c}\cdot\boldsymbol{\nabla} + \mathbf{a}\cdot\partial_{\mathbf{c}}, \tag{3.16}$$

where

$$\frac{D}{Dt} = \partial_t + \mathbf{u}\cdot\boldsymbol{\nabla} \tag{3.17}$$

and we use $\boldsymbol{\nabla}$ and $\partial_{\mathbf{x}}$ interchangeably. Then we have

$$f_{(1)} = -f_0\left[\mathcal{A} + \mathbf{c}\cdot\mathbf{B} + \frac{c^2}{2RT}\frac{D}{Dt}\ln T - (\mathbf{cc}{:}\boldsymbol{\nabla}\ln T)/(RT) + \frac{c^2}{2RT}\mathbf{c}\cdot\boldsymbol{\nabla}\ln T\right], \tag{3.18}$$

where

$$\mathcal{A} = \frac{D}{Dt}\ln\frac{\rho}{T^{3/2}}, \qquad \mathbf{B} = \boldsymbol{\nabla}\ln\frac{\rho}{T^{3/2}} + \left(\mathbf{a} - \frac{D\mathbf{u}}{Dt}\right)/(RT). \tag{3.19}$$

With the $\mathbf{c}$-related factors thus in evidence, it is a straightforward matter to carry out the necessary integrals to evaluate $\mathsf{P}_{(1)}$ and $\mathbf{Q}_{(1)}$. For this, we need the formula

$$\int e^i e^j e^k e^\ell \, d\Omega = \frac{4\pi}{15}(\delta^{ij}\delta^{k\ell} + \delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk}), \tag{3.20}$$

which may be verified by explicit evaluation of its components. If we include $\mathbb{P}_{(0)}$, we may write the result of the integration as

$$\mathsf{P} = \left[p - \mu\left(\frac{D\ln T}{Dt} + \frac{2}{3}\boldsymbol{\nabla}\cdot\mathbf{u}\right)\right]\mathsf{I} - \mu\mathsf{E} + O(\tau^2), \tag{3.21}$$

where $\mu = \tau p$, and

$$E^{ij} = \frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i} - \frac{2}{3}\boldsymbol{\nabla}\cdot\mathbf{u}\,\delta^{ij}. \tag{3.22}$$

For the heat current, we get

$$\mathbf{Q} = -\eta\boldsymbol{\nabla}T - \eta T\boldsymbol{\nabla}\ln p - \frac{5}{2}\mu\left(\frac{D\mathbf{u}}{Dt} - \mathbf{a}\right) + O(\tau^2), \tag{3.23}$$

where $\eta = \frac{5}{2}\mu R$.

A more detailed derivation is given in Ref. [26]. The expressions for P and $\mathbf{Q}$ involve not only the fluid dynamical fields, or lower moments of f, but their substantial derivatives as well. Those derivatives are given by the fluid dynamical equations that, in turn, involve P and $\mathbf{Q}$. So the higher moments are given only implicitly by Eqs. (3.22) and (3.23) and we need to solve the fluid equations together with those equations to obtain explicit expressions for the higher moments.

IV. DISCUSSION OF THE EQUATIONS

A. The entrance of entropy

Since the equations for P and $\mathbf{Q}$ are intertwined with the field equations themselves, we first rewrite these equations so as to clarify their meaning. We may introduce the continuity equation into Eq. (3.22) and so replace $\boldsymbol{\nabla}\cdot\mathbf{u}$ by $-\dot{\rho}/\rho$, where the dot stands for D/Dt. The combination of $\dot{T}/T$ and of $\dot{\rho}/\rho$ that then appears suggests the introduction of the quantity

$$S = \frac{3}{2}R\ln\frac{p}{\rho^{5/3}}. \tag{4.1}$$

For an ideal gas, this is the formula for the specific entropy, with $C_v = 3R/2$ and $\gamma = 5/3$. If we introduce S together with the definition of p, we may rewrite Eq. (3.21) as

$$\mathsf{P} = p\left[1 - \frac{\tau\dot{S}}{C_v}\right]\mathsf{I} - \mu\mathsf{E} + O(\tau^2). \tag{4.2}$$

Similarly, the momentum equation (3.2) can be used to rewrite the formula for the heat flux. Since Eq. (3.23) contains $D\mathbf{u}/Dt$, we may use Eq. (3.2) to rewrite it as

$$\mathbf{Q} = -\eta\boldsymbol{\nabla}T - \frac{5}{2}\nu\boldsymbol{\nabla}\cdot\left(\frac{\tau p\dot{S}}{C_v}\mathsf{I} + \tau p\mathsf{E}\right) + O(\tau^2), \tag{4.3}$$

where $\nu = \mu/\rho$.

We may also use Eq. (4.1) to convert Eq. (3.3) into an evolution equation for the specific entropy. If we introduce Eq. (4.2) into that equation, we find a term $p\boldsymbol{\nabla}\cdot\mathbf{u}$ so that $\dot{\rho}/\rho$ comes in by way of the continuity equation. The term in Eq. (3.3) involving $\dot{T}/T$ combines with this and we obtain an equation for $\dot{S}$. But $\dot{S}$ also appears in P and hence, when we gather the two apparitions of $\dot{S}$ together, we see that Eq. (3.3) becomes

$$\rho T\left(1 - \frac{2}{3}\tau\nabla\cdot\mathbf{u}\right)\frac{DS}{Dt} = -\mu\mathsf{E}{:}\boldsymbol{\nabla}\mathbf{u} - \boldsymbol{\nabla}\cdot\mathbf{Q}. \tag{4.4}$$

Though our equations may have an unfamiliar look, this is not because we have done anything unusual. Rather, we have omitted doing some things that are normally considered usual. So let us see how to get back to more familiar ground.

B. The Euler and Navier-Stokes equations

We have obtained

$$\mu = \tau p \quad \text{and} \quad \eta = \tfrac{5}{2}\tau p R \tag{4.5}$$

and so we see that when $\tau \to 0$, we have $\mathsf{P} = p\,\mathsf{I} + O(\tau)$ and $\mathbf{Q} = O(\tau)$, with $\dot{S} = O(\tau)$ according to Eq. (4.4). Thus we obtain the Euler equations in leading order together with entropy conservation.

We see also that the term $\tau\dot{S}/C_v$ in both Eq. (4.2) and Eq. (4.3) is of order τ^2. Therefore, we find that

$$\mathsf{P} = p\,\mathsf{I} - \mu\,\mathsf{E} + O(\tau^2) \tag{4.6}$$

and

$$\mathbf{Q} = -\eta \boldsymbol{\nabla} T + O(\tau^2), \tag{4.7}$$

when τ is very small. Therefore when the extra terms in our pressure tensor and heat flux are developed in τ, we see that our forms differ from the standard Navier-Stokes terms in terms of order τ^2, which is perfectly allowable in first-order theories.

In contrast to the conventional closure approximations, our expressions for P and $\mathbf{Q}$ depend on both the fluid fields and their derivatives. This means that these expressions must be solved in concert with the dynamical equations. To express P and $\mathbf{Q}$ explicitly in terms of the fluid fields, as in the usual closure relations, we would need to make expansions in τ. As we have just seen, in the first order, we recover the Euler equations and in the second order we get the Navier-Stokes equations. Continuation of this development produces terms of all orders in τ. Therein lies the crucial difference of our results from those of the Chapman-Enskog procedures. The extra terms in our development of the present results do not correspond to the higher theories based on Chapman-Enskog procedures that lead to the Burnett equations, as shall be explained in another place. All the terms in the present approximation come about from a first-order theory and the differences from standard theory arise in terms of second and higher order. When τ is not infinitesimal, which it never is in practice, these terms do have an effect on the predictions of the theory.

C. Dynamical pressure

To clarify the meaning of the difference between our equations for a simple gas and those obtained with the Chapman-Enskog procedure we note that, as in Eq. (3.6), we are writing the solution of the kinetic equation at any order as

$$f = f_N + O(\tau^{N+1}) \quad \text{with} \quad f_N = \sum_{m=0}^{N} f_{(m)}\tau^m. \tag{4.8}$$

From this we then obtain an approximation for the stress tensor in the form

$$\mathsf{P} = \mathsf{P}_N + \mathsf{R}_N, \tag{4.9}$$

where $\mathsf{R}_N = O(\tau^{N+1})$ is the error incurred in the truncation of the series. If we take the trace of Eq. (4.9), we find

$$\mathrm{Tr}\,\mathsf{P} = \mathrm{Tr}\,\mathsf{P}_N + \mathrm{Tr}\,\mathsf{R}_N. \tag{4.10}$$

As we see from the definition (3.4) of P, in the exact case,

$$\mathrm{Tr}\,\mathsf{P} = 3p. \tag{4.11}$$

This result says that the total pressure has no contribution of a dynamical kind for a structureless gas, as Maxwell and Boltzmann both recognized. The Chapman-Enskog procedure imposes this condition at every finite order so that

$$\mathrm{Tr}\,\mathsf{P}_N^{CE} = 3p. \tag{4.12}$$

When we impose Eq. (4.12) onto Eq. (4.10), we are forcing the requirement that $\mathrm{tr}\,\mathsf{R}_N = 0$, which overly constrains the results obtained by Chapman and Enskog and destroys any hope of improving convergence by mitigating their errors. However, all that we should demand of our successive approximations is that in the Nth approximation,

$$\mathrm{Tr}\,\mathsf{P}_N = 3p + O(\tau^{N+1}). \tag{4.13}$$

On introducing this less restrictive condition into Eq. (4.10) we find that, in our procedure,

$$\mathrm{Tr}\,\mathsf{R}_N = O(\tau^{N+1}), \tag{4.14}$$

which does not incur the loss of generality that forcing the trace of R_N to vanish does. By keeping terms $O(\tau^{N+1})$ in our Nth approximation for P and $\mathbf{Q}$, we leave open the possibility of compensating for the errors caused by the truncation of the series for f by retaining suitable process-dependent effects of higher order.

Thus we have in our present approximation a dynamical pressure in our approximate pressure tensor, $\mathsf{P}_1 = \mathsf{P}_{(0)} + \tau\mathsf{P}_{(1)}$, that is,

$$\mathrm{Tr}\,\mathsf{P}_1 = 3p\left[1 - \frac{\tau\dot{S}}{C_v}\right] = 3p + O(\tau^2). \tag{4.15}$$

Our asymptotics suggests that the extra term compensates for the effect of truncation of the series for f. Since this term has contributions from all orders in τ, it can in principle be very effective in extending the domain of validity of the theory.

V. CONCLUSION

We have illustrated our derivation of the fluid equations from kinetic equations by carrying out the procedure for the relaxation model of kinetic theory. The same procedure can be used on other forms of the kinetic equation. In the case of the Boltzmann equation, the procedure is quite similar, though the inversion of the linearized Boltzmann collision operator involves some technical issues that we shall take up elsewhere. However, by working out the case of the relaxation model we can more readily see the differences between our approach and the Chapman-Enskog procedure.

In our derivation, we do not introduce slow times as in the Chapman-Enskog method. This means that we are not driven to expand the fluid variables (or slow quantities) in τ as in Chapman-Enskog theory. Such expansions cause ambiguity in the application of initial conditions since it is not clear how to distribute the initial values over the various orders. Moreover, those expansions lead to a different sequence of approximations than ours. Chapman-Enskog theory gives the pressure tensor and the heat flux explicitly in terms of the fluid fields, a feature that results from invoking solvability conditions at each order. Our results do not produce explicit formulas for $\mathbb{P}$ and **Q** in terms of the fluid fields; rather, these quantities are expressed in terms of the fluid fields *and* their derivatives. Those derivatives appear in the field equations themselves so that we do produce a closed system of equations. Moreover, since the expansion variable τ appears in the equations, we may further expand the equations to develop explicit formulas for $\mathbb{P}$ and **Q** accurate to any prescribed order.

As we saw, the leading terms in the development of the formula for $\mathbb{P}$ and **Q** give us successively the Euler and Navier-Stokes equations, but the development need not stop there. That is, our finite formulas implicitly contain terms of all orders in τ and this means that we may hope that they will produce high accuracy even when τ is not infinitesimal. In any event, it is clear that we may expect a divergence between results from our system and those from the Navier-Stokes equations when τ is not very small.

Keller [27] has remarked that two theories with the same nominal accuracy may have different domains of validity. In forcing the trace of $\mathbb{P}_N$ to be exactly $3p$, order by order, the Chapman-Enskog method renounces the extra generality allowed by the freedom to choose higher order terms in an advantageous way. For us, the problem has been to select the best way to allow for the higher order corrections when trying to extend the domain of validity of the theory. We shall use as our test of validity the comparison with experiment given in the following paper of this series.

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